ON OPERATORS WHICH ATTAIN THEIR NORM

BY

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ABSTRACT

The following problem is considered. Let X and Y be Banach spaces. Are those operators from X to Y which attain their norm on the unit cell of X, norm dense in the space of all operators from X to Y? It is proved that this is always the case if X is reflexive. In general the answer is negative and it depends on some convexity and smoothness properties of the unit cells in X and Y. As an application a refinement of the Krein-Milman theorem and Mazur's theorem concerning the density of smooth points, in the case of weakly compact sets in a separable space, is obtained.

1. **Introduction.** Let $B(X, Y)$ be the Banach space of all bounded linear operators from the Banach space X into the Banach space Y. (The norm in $B(X, Y)$ is the usual operator norm.) Let $P(X, Y)$ be the subset of $B(X, Y)$ consisting of all the operators which attain their norm on the unit cell of X, that is all those T for which there is an $x \in X$ satisfying $\| x \| = 1$ and $\| Tx \| = \| T \|$. Bishop and Phelps [1] (cf. also [2]) proved that if dim $Y = 1$ then $P(X, Y)$ is norm dense in $B(X, Y)$ for every Banach space X. In [1] they also raised the general question—for which Banach spaces X and Y is $P(X, Y)$ norm dense in $B(X, Y)$? This question is the subject of the present note.

Rather simple examples show that in general $P(X, Y)$ is not dense in $B(X, Y)$. The simplest examples, perhaps, are based on the fact that if a $1-1$ operator T from X into a strictly convex space Y attains its norm at a point x, then x is an extreme point of the unit cell of X. However, if we consider instead of $P(X, Y)$ the larger set $P_0(X, Y)$, consisting of all the operators T such that T^{**} attains its norm on the unit cell of X^{**}, then it can be shown (Theorem 1) that this set is always norm dense in $B(X, Y)$.

The question of Bishop and Phelps, as it stands, is very general. In fact, it seems to be too general to have a reasonably complete solution. We therefore restrict ourselves here to the study of those spaces X which have either one of the following properties.

A. For every Banach space Y, $P(X, Y)$ is norm dense in $B(X, Y)$.

B. For every Banach space Y, $P(Y, X)$ is norm dense in $B(Y, X)$.

An immediate consequence of the density of $P_0(X, Y)$ in $B(X, Y)$ is that every

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reflexive space has property A. In Theorem 2 we show that if a Banach space X has property A then, under certain circumstances (for example if it is separable), its unit cell must have many strongly exposed points(2). A dual result concerning property B and strongly smooth points is also given (Theorem 3). As a consequence of the results mentioned above we obtain (for weakly compact sets in a separable space) refinements of the Krein-Milman Theorem and Mazur's Theorem [10] concerning the density of smooth points (cf. Theorem 4).

In section 3 we prove some results of a more special nature and discuss a few simple examples. It is shown in particular that a finite-dimensional space whose unit cell is a polyhedron has property A and that there are Banach spaces X such that $P(X,X)$ is not norm dense in $B(X,X)$.

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Notations. By "operator" we always mean a bounded linear operator. Our results hold for real and for complex Banach spaces; however, for convenience of notation, we shall assume that the Banach spaces are real. The unit cell $\{x: x \in X, \| x \| \leqq 1\}$ of X is denoted by S_X. Let C be a convex set in the Banach space X. A point $x \in C$ is called an *exposed point* of C if there is an $f \in X^*$ such that $f(y) < f(x)$ for every $y \neq x$ in C. A point $x \in C$ is called a *strongly exposed point* of C if there is an $f \in X^*$ such that (i) $f(y) < f(x)$ for $y \neq x$ in C; and (ii) $f(x_n) \to f(x)$ and $\{x_n\}_{n=1}^{\infty} \subset C$ imply $\| x_n - x \| \to 0$. The dual notions are those of a smooth and strongly smooth point. We shall use these notions only for the unit cell and so we define them only in this case. A point x with $\| x \| = 1$ is called a *smooth point* of S_X if there exists only one $f \in X^*$ satisfying $f(x) = \| f \| = 1$. A point $x \in X$ with $\| x \| = 1$ is called a *strongly smooth point* of S_X if $f_n(x) \to 1$, and $\{f_n\}_{n=1}^{\infty} \subset S_{X^*}$ imply that $\| f_n - f \| \to 0$ (where f is, necessarily, the unique element of S_{X^*} satisfying $f(x) = 1$). A point x with $\| x \| = 1$ is a smooth point (resp. strongly smooth point) of S_X if and only if the norm is Gateaux (resp. Fréchet) differentiable at x (cf. Šmul'yan [11]). A Banach space X is called strictly convex (resp. smooth) if every point on the boundary of S_X is an exposed (resp. smooth) point of S_X. X is called locally uniformly convex [9] if $\| x_n + x \| \to 2$, and $\| x_n \| = \| x \| = 1$ imply $\| x_n - x \| \to 0$. If X is locally uniformly convex then every point on the boundary of S_X is a strongly exposed point. If every point on the boundary of S_X is strongly smooth we say that the norm in X is Fréchet differentiable (cf. Day [5, pp. 112-113] for these and related notions).

2. **The main results.** We begin by giving a simple characterization of the set $P_0(X, Y)$.

LEMMA 1. *An operator T from X to Y belongs to $P_0(X,Y)$ if and only if there are $\{x_k\}_{k=1}^{\infty}$ in X and $\{f_k\}_{k=1}^{\infty}$ in Y^* such that*

(2) This notion is defined below.

$$\| x_k \| = \| f_k \| = 1 \qquad k = 1,2,\cdots \tag{1}$$

$$|f_j(Tx_k)| \geqq \| T\| - 1/j \qquad j \leqq k, \quad k = 1,2,\cdots \tag{2}$$

Proof. Suppose (1) and (2) hold and let x^{**} be any weak* limit point of the sequence $\{x_k\}_{k=1}^{\infty}$. Then, for every j, $| T^{**}x^{**}(f_j)| \geqq \| T\| - 1/j$ and hence $\| T^{**}x^{**} \| = \| T^{**} \|$. The proof of the converse is also immediate (using the weak* density of S_X in $S_{X^{**}}$).

THEOREM 1. *For every X and Y, $P_0(X, Y)$ is norm dense in $B(X, Y)$. Hence every reflexive space has property A.*

Proof. Let $T \in B(X, Y)$ with $\| T \| = 1$ and an ε with $0 < \varepsilon < 1/3$ be given. We choose first a monotonically decreasing sequence $\{\varepsilon_k\}$ of positive numbers such that

$$2\sum_{i=1}^{\infty} \varepsilon_i < \varepsilon, \quad 2 \sum_{i=k+1}^{\infty} \varepsilon_i < \varepsilon_k^2, \quad \varepsilon_k < 1/10k, \qquad k = 1,2,\cdots \tag{3}$$

We next choose inductively sequences $\{T_k\}_{k=1}^{\infty}$, $\{x_k\}_{k=1}^{\infty}$, and $\{f_k\}_{k=1}^{\infty}$ satisfying

$$T_1 = T \tag{4}$$

$$\| T_k x_k \| \geqq \| T_k \| - \varepsilon_k^2, \ \| x_k \| = 1 \qquad k = 1,2,\cdots \tag{5}$$

$$f_k(T_k x_k) = \| T_k x_k \|, \ \| f_k \| = 1 \qquad k = 1,2,\cdots \tag{6}$$

$$T_{k+1}x = T_k x + \varepsilon_k f_k(T_k x) \cdot T_k x_k \qquad x \in X, \quad k = 1,2,\cdots \tag{7}$$

Having chosen these sequences we verify that the following hold.

$$\| T_j - T_k \| \leqq 2\sum_{i=j}^{k-1} \varepsilon_i, \quad \| T_k \| \leqq 4/3 \qquad j < k, \quad k = 1,2,\cdots \tag{8}$$

$$\| T_{k+1} \| \geqq \| T_k \| + \varepsilon_k \| T_k \|^2 - 4\varepsilon_k^2 \qquad k = 1,2,\cdots \tag{9}$$

$$\| T_k \| \geqq \| T_j \| \geqq 1 \qquad j < k, \quad k = 1,2,\cdots \tag{10}$$

$$|f_j(T_j x_k)| \geqq \| T_j \| - 6\varepsilon_j \qquad j < k, \quad k = 1,2,\cdots \tag{11}$$

Assertion (8) is easily proved by using induction on k. By (5), (6) and (7)

$$\| T_{k+1} \| \geqq \| T_{k+1}x_k \| = \| T_k x_k(1 + \varepsilon_k f_k(T_k x_k)) \|$$

$$= \| T_k x_k \|(1 + \varepsilon_k\| T_k x_k \|) \geqq (\|T_k \| - \varepsilon_k^2)(1 + \varepsilon_k\| T_k \| - \varepsilon_k^3).$$

Relation (9) follows easily from this inequality, since $\| T_k \| \leqq 4/3$ and $\varepsilon_k < 1/10k$, while (10) is an immediate consequence of (4) and (9). Finally we verify (11). By the triangle inequality, (5), (8), (10) and (3) we have, for $j < k$,

$$\| T_{j+1}x_k \| \geqq \| T_k x_k \| - \| T_k - T_{j+1} \| \geqq$$

$$\geqq \| T_k \| - \varepsilon_k^2 - 2 \sum_{i=j+1}^{k-1} \varepsilon_i \geqq \| T_{j+1} \| - 2\varepsilon_j^2.$$

Hence, by (7) and (9),

$$\varepsilon_j | f_j(T_j x_k) | \, \| T_j \| + \| T_j \| \geqq \| T_{j+1} x_k \| \geqq$$

$$\geqq \| T_j \| + \varepsilon_j \| T_j \|^2 - 6\varepsilon_j^2,$$

so that

$$| f_j(T_j x_k) | \geqq \| T_j \| - 6\varepsilon_j,$$

and this proves (11).

The sequence T_k converges in norm to an operator $\hat{T}$ satisfying $\| \hat{T} - T \| \leqq \varepsilon$ and $\| \hat{T} - T_j \| \leqq \varepsilon_{j-1}^2$ for $j = 2, 3, \cdots$ (use (3) and (8)). We claim that $\hat{T} \in P_0(X, Y)$. Indeed,

$$| f_j(\hat{T} x_k) | \geqq | f_j(T_j x_k) | - \| T_j - \hat{T} \| \geqq$$

$$\geqq \| T_j \| - 6\varepsilon_j - \varepsilon_{j-1}^2 \geqq \| \hat{T} \| - 6\varepsilon_j - 2\varepsilon_{j-1}^2 \geqq \| \hat{T} \| - 1/j,$$

and the desired conclusion follows from Lemma 1. For reflexive X it is obvious that $P(X, Y) = P_0(X, Y)$ and thus every reflexive X has property A.

Remark. The operator $\hat{T}$ constructed in the proof of Theorem 1 has also the property that $\hat{T} - T$ is compact.

THEOREM 2. *Let the Banach space X have property A. Then*

(i) *If X is isomorphic to a strictly convex space, then S_X is the closed convex hull of its exposed points.*

(ii) *If X is isomorphic to a locally uniformly convex space, then S_X is the closed convex hull of its strongly exposed points.*

Proof. The proofs of (i) and (ii) are almost identical so we prove here only (ii). Let C be the closed convex hull of the strongly exposed points of S_X. Suppose that $C \neq S_X$. Then there is an $f \in X^*$ with $\| f \| = 1$ and a $\delta > 0$ such that $| f(x) | < 1 - \delta$ for $x \in C$. Let $||| \quad |||$ be a locally uniformly convex norm in X which is equivalent to the given norm $\| \quad \|$ and such that $||| x ||| \leqq \| x \|$ for every x. Let Y be the space $X \oplus R$(3) with the norm $\| (x, r) \| = (||| x |||^2 + r^2)^{1/2}$. Then Y is locally uniformly convex. Let V be the operator from X into Y defined by $Vx = (x, Mf(x))$ where $M > 2/\delta$. Then V is an isomorphism (into) and the same is true for every operator sufficiently close to V. We have

$$\| V \| \geqq M; \ \| Vx \| \leqq (1 + (M-2)^2)^{1/2} \leqq M - 1 \text{ for } x \in C.$$

(3) R denotes the one-dimensional space.

It follows that operators sufficiently close to V cannot attain their norm at a point belonging to C. To conclude the proof we have only to show that if T is an isomorphism (into) which attains its norm at a point x and if the range of T is locally uniformly convex, then x is a strongly exposed point of S_X. Indeed, let $g \in Y^*$ satisfy $\| g \| = 1$, $g(Tx) = \| T \|$. Suppose that $g(Tx_n) \to \| T \|$, with $\| x_n \| \leqq 1$. Then $\| Tx + Tx_n \| \to 2\| T \|$ and hence, by our assumption on Y, $\| Tx - Tx_n\| \to 0$. Since T is an isomorphism, $\| x_n - x \| \to 0$, and this concludes the proof of the theorem.

For examples of X which satisfy the assumption in (i) cf. Day [3], [4]. Kadec [6] has proved that every separable Banach space is isomorphic to a locally uniformly convex space.

We prove next a partial converse to Theorem 2(ii). We say that a family of points $\{x_\alpha\}$ on the boundary of S_X is *uniformly strongly exposed* (u.s.e.) if there is a function $\delta(\varepsilon)$, with $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$, and a set $\{f_\alpha\}$ of elements of norm 1 in X^* such that for every α, $f_\alpha(x_\alpha) = 1$, and for any x,

$$\| x \| \leqq 1 \text{ and } f_\alpha(x) \geqq 1 - \delta(\varepsilon) \text{ imply } \| x - x_\alpha \| \leqq \varepsilon.$$

In a uniformly convex space the set of all the boundary points of the unit cell is u.s.e. The set of all the extreme points of the unit cell of l_1 is also u.s.e.

PROPOSITION 1. *Suppose S_X is the closed convex hull of a set of uniformly strongly exposed points. Then X has property A.*

Proof. The proof is similar to that of Theorem 1 and we indicate here only the necessary modifications. Let $\{x_\alpha\}$ be a set of u.s.e. points whose closed convex hull is S_X and let $\{f_\alpha\}$ be the corresponding set in X^* (appearing in the definition of a u.s.e. set). We choose the ε_i as in the proof of Theorem 1 and define a sequence of operators T_k as follows. $T_1 = T$ and

$$T_{k+1}x = T_k x + \varepsilon_k f_{\alpha_k}(x) \cdot T_k x_{\alpha_k} \qquad k = 1,2,\cdots$$

where x_{α_k} is an element of $\{x_\alpha\}$ satisfying $\| T_k x_{\alpha_k} \| \geqq \| T_k \| - \varepsilon_k^2$, and f_{α_k} is the corresponding element of $\{f_\alpha\}$. As in the proof of Theorem 1 it can be shown that the sequence T_k converges in the norm topology to an operator $\hat{T}$ satisfying $\| \hat{T} - T \| \leqq \varepsilon$, and that $| f_{\alpha_j}(x_{\alpha_k}) | > 1 - 1/j$ for $j < k$. By the definition of a u.s.e. set it follows that the sequence x_{α_k} converges in the norm topology to a point x, say, and we have $\| \hat{T}x \| = \| \hat{T} \|$. This concludes the proof.

REMARK. There exist even finite-dimensional spaces whose unit cells cannot be obtained as the closed convex hull of a u.s.e. set. Indeed, in a finite-dimensional space the closure of a u.s.e. set is again u.s.e., and simple 2-dimensional examples show that in general a convex set cannot be obtained as the closed convex hull of a closed subset of its set of exposed points.

Our next result is concerned with some smoothness properties.

THEOREM 3. *Let X have property B. Then*

(i) *If X is isomorphic to a quotient space of a smooth space then the smooth points of S_X are norm dense in the boundary of S_X.*

(ii) *If X is isomorphic to a quotient space of a space which has a Fréchet differentiable norm, then the strongly smooth points of S_X are norm dense in the boundary of S_X.*

Proof. Again the proofs of parts (i) and (ii) are almost identical so we prove only (ii). Let Z be a space whose norm is Fréchet differentiable and let T_0 be an operator from Z onto X with $\| T_0 \| \leqq 1$. Let Y be $Z \oplus R$ with $\| (z, r) \| = (\| z \|^2 + r^2)^{1/2}$. The norm in Y is also Fréchet differentiable. Take x_0 on the boundary of S_X and define $T \in B(Y,X)$ by $T(z,r) = T_0 z + rMx_0$, where M is a sufficiently large positive number. Let $\varepsilon > 0$ be given and let $\hat{T} \in P(Y,X)$ satisfy $\| \hat{T} - T \| \leqq \varepsilon$. Since T^* is an isomorphism into, the same is true for $\hat{T}^*$ if ε is small enough. Take $y_0 = (z_0, r_0) \in Y$ with $\| y_0 \| = 1$ and $\| \hat{T} y_0 \| = \| \hat{T} \|$. We have $\hat{T} y_0 = T_0 z_0 + r_0 M x_0 + u$ for some u of norm $\leqq \varepsilon$, and $\| \hat{T} y_0 \| \geqq M - \varepsilon$. Put $x_1 = \hat{T} y_0 / \| \hat{T} \|$. Since $\| T_0 z_0 \| \leqq 1$ it follows easily that $\| x_0 - x_1 \| = O(M^{-1})$ as $M \to \infty$ (uniformly with respect to ε in $[0,1]$)([3a]). Hence to conclude the proof it is sufficient to prove that x_1 is a strongly smooth point of S_X. Take an $f \in X^*$ with $f(x_1) = \| f \| = 1$ and let $f_n(x_1) \to 1$ with $\| f_n \| \leqq 1$. Put $g = \hat{T}^* f / \| \hat{T} \|$ and $g_n = \hat{T}^* f_n / \| \hat{T} \|$. Clearly $g(y_0) = \| g \| = 1$ and $g_n(y_0) \to 1$. Since the norm in Y is Fréchet differentiable $\| g_n - g \| \to 0$ and hence $\| f_n - f \| \to 0$ ($\hat{T}^*$ is an isomorphism). This concludes the proof.

Our main reasons for stating and proving Theorem 3 are its obvious duality with Theorem 2 and the application of its proof in Theorem 4b. In contrast to Theorem 2, Theorem 3 does not seem to be a useful criterion for deciding whether a given Banach space has property B. This is because the assumptions in (i) or (ii) here hold less frequently than the corresponding ones in Theorem 2 while the conclusion of (i), for example, holds for every separable space (cf. Day [3] and Klee [8] for more details). It is even conceivable that statement (i) in Theorem 3 holds in general, that is, without the assumption that X has property B. Concerning statement (ii) it should be remarked, perhaps, that if a Banach space X has a Fréchet differentiable norm then the density character of X^* is the same as that of X. This result was recently announced by Kadec [7] and it is a consequence of the theorem of Bishop and Phelps.

The results which we have already proved imply easily the following refinement of the Krein-Milman theorem and the density theorem of Mazur [10] (cf. also Klee [8]).

THEOREM 4. *a. Every weakly compact convex set in a separable Banach space is the closed convex hull of its strongly exposed points.*

([3a]) Here we assume that $r_0 \geqq 0$. If $r_0 < 0$ we replace y_0 by $-y_0$.

b. The boundary of the unit cell of a separable reflexive space has a dense set of strongly smooth points.

Proof. From Theorems 1 and 2 it follows immediately that the unit cell of a separable reflexive space is the closed convex hull of its strongly exposed points. The more general statement of (a) results from the observation that the method of proof of Theorem 1 yields the following statement: For every X and Y the set consisting of all the operators T which attain their supremum on a given weakly compact convex set C in X (that is, for which there exists an $x_0 \in C$ with $\| Tx_0 \| = \sup_{x \in C} \| Tx \|$) is norm dense in $B(X, Y)$. Theorem 2 can also be modified in an obvious way so as to apply to operators which attain their supremum on a fixed C.

Part (b) follows by observing that if X is reflexive and separable, we can take as Y in the proof of Theorem 3 a space isomorphic to $X \oplus R$. Since Y is reflexive the density of $P(Y,X)$ in $B(Y,X)$ follows from Theorem 1 (and hence we do not use the assumption in Theorem 3 that X has property B).

Remark. There exist separable reflexive spaces whose unit cells have exposed points which are not strongly exposed. For example, let X be the space l_2 and denote by S its unit cell in the usual norm. Let $e_n = (1 - 1/n, 0, \cdots, 0, 1, 0, \cdots)$ for $n = 2,3,\cdots$ (the number 1 stands in the n-th place). Let S_1 be the closed convex hull of $\bigcup_{n=2}^{\infty} \{\pm e_n\} \cup S$. It is easy to see that the point $(1,0,0,\cdots)$ is the only point in S_1 whose first coordinate is 1 and hence it is an exposed point of S_1. It is, however, not strongly exposed since the first coordinate of e_n tends to 1. It follows, by duality, that there are separable reflexive spaces whose unit cells have smooth boundary points which are not strongly smooth. It should be remarked also that there exist separable (non-reflexive) Banach spaces whose unit cells do not have any strongly smooth boundary point (l_1, for example).

3. **Some examples.** Using known representation theorems for operators it is possible in some special cases to verify directly the density of $P(X, Y)$ in $B(X, Y)$. We shall now give a few results in which some of the common spaces are examined as to whether they have property A or B. The results are, however, very incomplete. Even for the spaces $C(K)$ we do not know a complete characterization of those which have either property A or B. Moreover, the examples given here do not answer some questions which naturally arise in connection with the results proved in section 2 (for example: Does every reflexive space have property B?).

Proposition 2. *a. The space $L_1(\mu)$ has property A if and only if the measure μ is purely atomic.*

b. The space $C(K)$ with K compact metric has property A if and only if K is a finite set.

Proof. It is well known and easy to see that the extreme points of the unit cell of an $L_1(\mu)$ space are the characteristic functions of the atoms of μ (multiplied by a suitable scalar so as to have norm 1). Hence the unit cell of $L_1(\mu)$ is the closed convex hull of its extreme points if and only if μ is purely atomic. Every L_1 space is isomorphic to a strictly convex space (Day [4]). Hence, by Theorem 2(i), $L_1(\mu)$ does not have property A if μ is not purely atomic. That an $l_1(I)$ space has property A for every index set I is very easily seen directly (it follows also from Proposition 1).

To prove part (b) it is, by Theorem 2(ii), sufficient to show that if K is infinite compact Hausdorff, then the unit cell of $C(K)$ has no strongly exposed point. Suppose there were such a point x. Let μ be the functional(4) appearing in the definition of a strongly exposed point and let $\varepsilon > 0$. There exists a non-void open set G in K such that $0 \leqslant |\mu|(G) < \varepsilon/2$. Since x is an extreme point of the unit cell of $C(K)$, $|x(k)| = 1$ for every $k \in K$. It is now easily seen that there is a $y \in C(K)$ such that $\| y \| \leqq 1$, $\| y - x \| = 2$ and $y(k) = x(k)$ for $k \notin G$. Clearly $\mu(y) \geqq 1 - \varepsilon$ and this contradicts the definition of a strongly exposed point.

Our next result is a consequence of the theorem of Bishop and Phelps [1]. It may be regarded as the dual of Proposition 1.

PROPOSITION 3. *Let X be a Banach space such that there exist two sets $\{x_\alpha\}$ in X and $\{f_\alpha\}$ in X^*(5) and $\lambda < 1$ such that*

1. $\|f_\alpha\| = 1$ *for every* α *and* $\| x \| = \sup_\alpha |f_\alpha(x)|$ *for every* $x \in X$.
2. $\| x_\alpha \| = f_\alpha(x_\alpha) = 1$ *for every* α *and* $|f_\alpha(x_\beta)| < \lambda$ *for* $\alpha \neq \beta$.

Then X has property B.

Proof. Let $T \in B(Y,X)$ with $\| T \| = 1$ and ε, $0 < \varepsilon < 1$, be given. Clearly $1 = \| T \| = \sup_\alpha \| T^* f_\alpha \|$. Let α_0 be such that $\| T^* f_{\alpha_0} \| \geqq 1 - \varepsilon(1-\lambda)/4$. Choose a $g \in Y^*$ which attains its norm on S_Y and satisfies

$$\| g - T^* f_{\alpha_0} \| \leqq \varepsilon(1-\lambda)/2, \quad 1 - \varepsilon(1-\lambda)/4 \leqq \| g \| \leqq 1.$$

Put

$$T_0 y = Ty + [(1+\varepsilon)g(y) - T^* f_{\alpha_0}(y)]x_{\alpha_0}.$$

We have

$$\| T - T_0 \| \leqq \varepsilon \| g \| + \| T^* f_\alpha - g \| \leqq 2\varepsilon.$$

Further, $T_0 \in P(Y,X)$. Indeed, for $\alpha \neq \alpha_0$,

$$\| T_0^* f_\alpha \| \leqq \| T \| + |f_\alpha(x_{\alpha_0})| (\varepsilon \| g \| + \| T^* f_{\alpha_0} - g \|) \leqq$$
$$\leqq 1 + \lambda(\varepsilon + \varepsilon(1-\lambda)/2) \leqq 1 + \varepsilon(1+\lambda)/2.$$

(4) We identify the functional with the corresponding measure on K.

(5) In both sets the set of indices $\{\alpha\}$ is the same.

while $T_0{}^*f_{\alpha_0} = (1 + \varepsilon)g$ satisfies

$$\| T_0{}^*f_{\alpha_0} \| = (1 + \varepsilon)\| g \| \geqq (1 + \varepsilon)(1 - \varepsilon(1 - \lambda)/4) \geqq 1 + \varepsilon(1 + \lambda)/2.$$

Hence $\| T_0{}^* \| = \| T_0{}^*f_{\alpha_0} \|$ and since $T_0^*f_{\alpha_0}$ attains its norm on S_Y the same is true for T_0.

The assumptions in Proposition 3 are satisfied if, for example, X is finite-dimensional and its unit cell is a polyhedron or if $X = C(K)$ with K having a dense set of isolated points.

Many examples of spaces which do not have property B can be obtained by using Theorem 2.

For example, if X is strictly convex and if there is a Banach space Y such that S_Y is not the closed convex hull of its exposed points and such that Y is isomorphic to a proper subspace of X then X does not have property B.

In some special cases it is easy to obtain somewhat stronger results. We have for example

PROPOSITION 4. *If X is strictly convex and if there is a non-compact operator from c_0 into X then X does not have property* B.

Proof. Suppose $T \in P(c_0, X)$ and let $y \in c_0$ satisfy $\| y \| = 1$ and $\| T_y \| = \| T \|$. Denote by $\{e_i\}_{i=1}^{\infty}$ the natural basis of c_0. There is an integer n such that for $i > n$ $\| y \pm e_i/2 \| = 1$. It follows that for these i, $\| T_y \pm Te_i/2 \| \leqq \| T_y \|$ and hence, by the strict convexity of X, $Te_i = 0$ for $i > n$. Thus every operator belonging to $P(c_0, X)$ has a finite-dimensional range and, as a consequence, every operator in the closure of $P(c_0, X)$ is compact.

Finally we observe the following.

PROPOSITION 5. *There exist Banach spaces X for which $P(X, X)$ is not dense in $B(X, X)$.*

Proof. Let $Y = c_0$ with the usual norm and let Z be a strictly convex space isomorphic to c_0. Put $X = Y \oplus Z$ with $\| (y, z) \| = \max(\| y \|, \| z \|)$(6). X has the required property. Indeed, let T_0 be an isomorphism from Y onto Z with $\| T_0 \| \leqq 1$ and define T in $B(X, X)$ by $T(y, z) = (0, T_0 y)$. We have $\| T_0 y \| \geqq 2\varepsilon \| y \|$ for every $y \in Y$ and some $\varepsilon > 0$. Suppose there were a $\hat{T} \in B(X, X)$ with $\| \hat{T} - T \| < \varepsilon$ and $\| \hat{T} \| = \| \hat{T}(y_0, z_0) \|$ for some (y_0, z_0) in X of norm 1. Put $\hat{T}(y_0, z_0) = (u, v)$. Clearly $\| u \| < \varepsilon$ and since $\| \hat{T} \| > \varepsilon$ it follows that $\| u \| < \| \hat{T} \| = \| v \|$. Since S_Y has no extreme point there is a $y_1 \neq 0$ in Y such that

$$\| y_1 + y_0 \| = \| - y_1 + y_0 \| \leqq 1.$$

(6) For every vector it will be clear to which space it belongs. Therefore we use the same notation, $\| \; \|$, for the norms in X, Y and Z.

Hence

$$\| \hat{T}(y_0, z_0) \pm \hat{T}(y_1, 0)\| \leqq \| \hat{T} \|.$$

Since Z is strictly convex and $\| v \| = \| \hat{T} \|$ it follows that $\hat{T}(y_1, 0) = (y_2, 0)$ for some $y_2 \in Y$. We get

$$\varepsilon \| y_1 \| \geqq \| T(y_1, 0) - \hat{T}(y_1, 0) \| \geqq \| T_0 y_1 \| \geqq 2\varepsilon \| y_1 \|$$

and this is a contradiction.

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